Partial Reductions for Kleene Algebra with Linear Hypotheses

Liam Chung, Tobias Kappé

14th April, 2026

LIACS, Leiden University
FoSSaCS 2026

These slides on my website:

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Partially Decidable Program Equivalence

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...with Kleene Algebra

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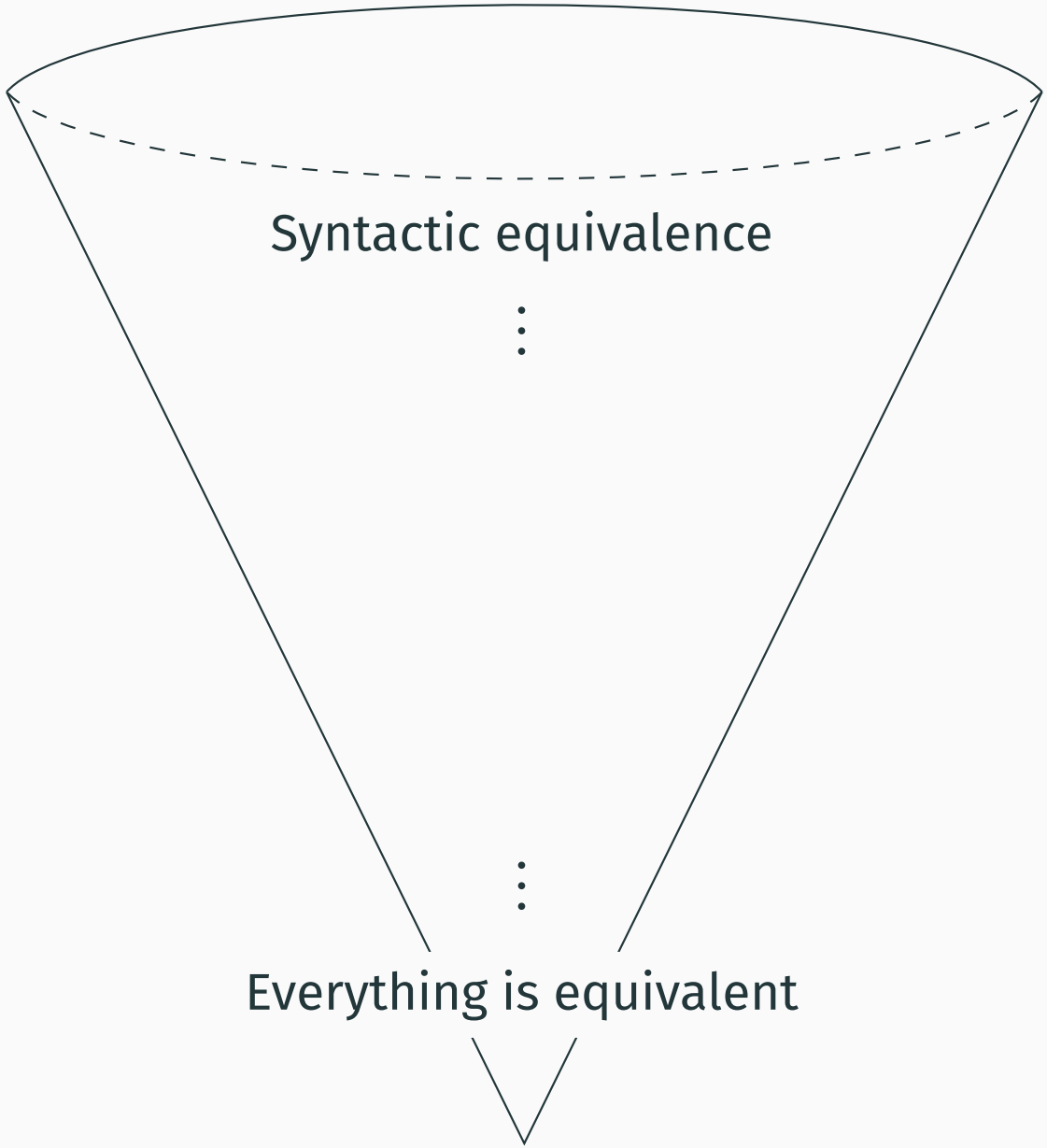
A Top-Down View on Program Equivalence Semantics



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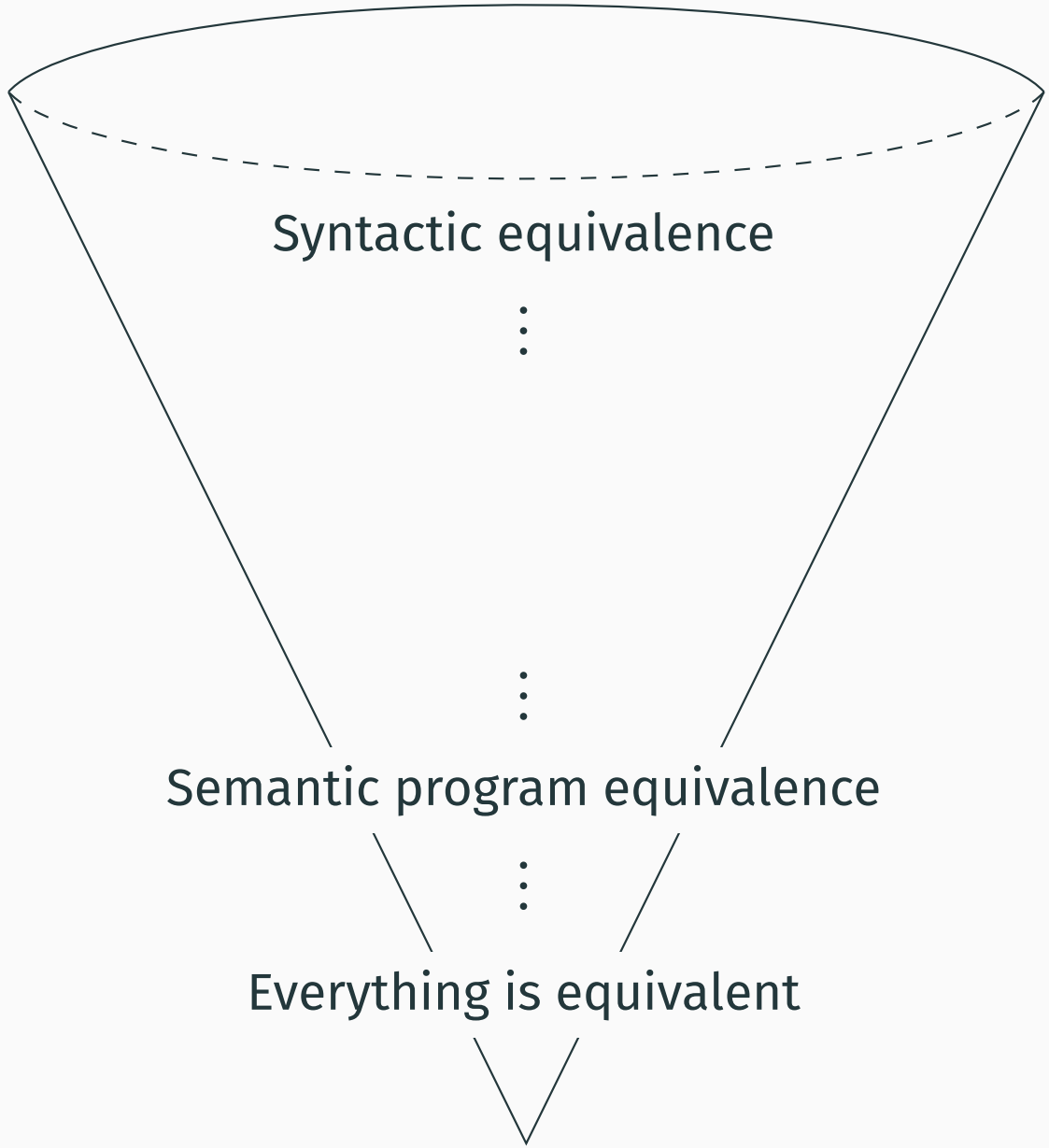

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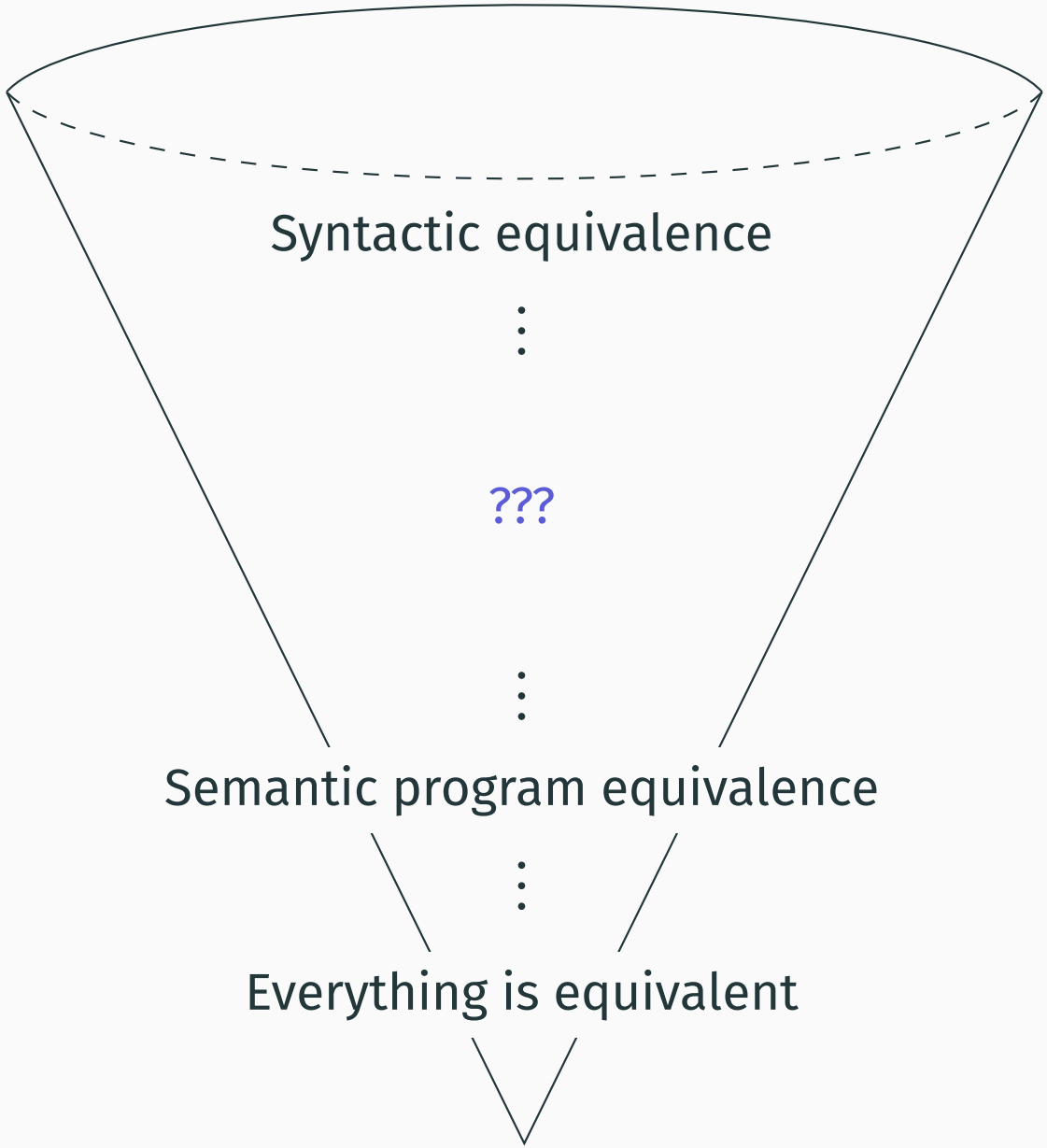

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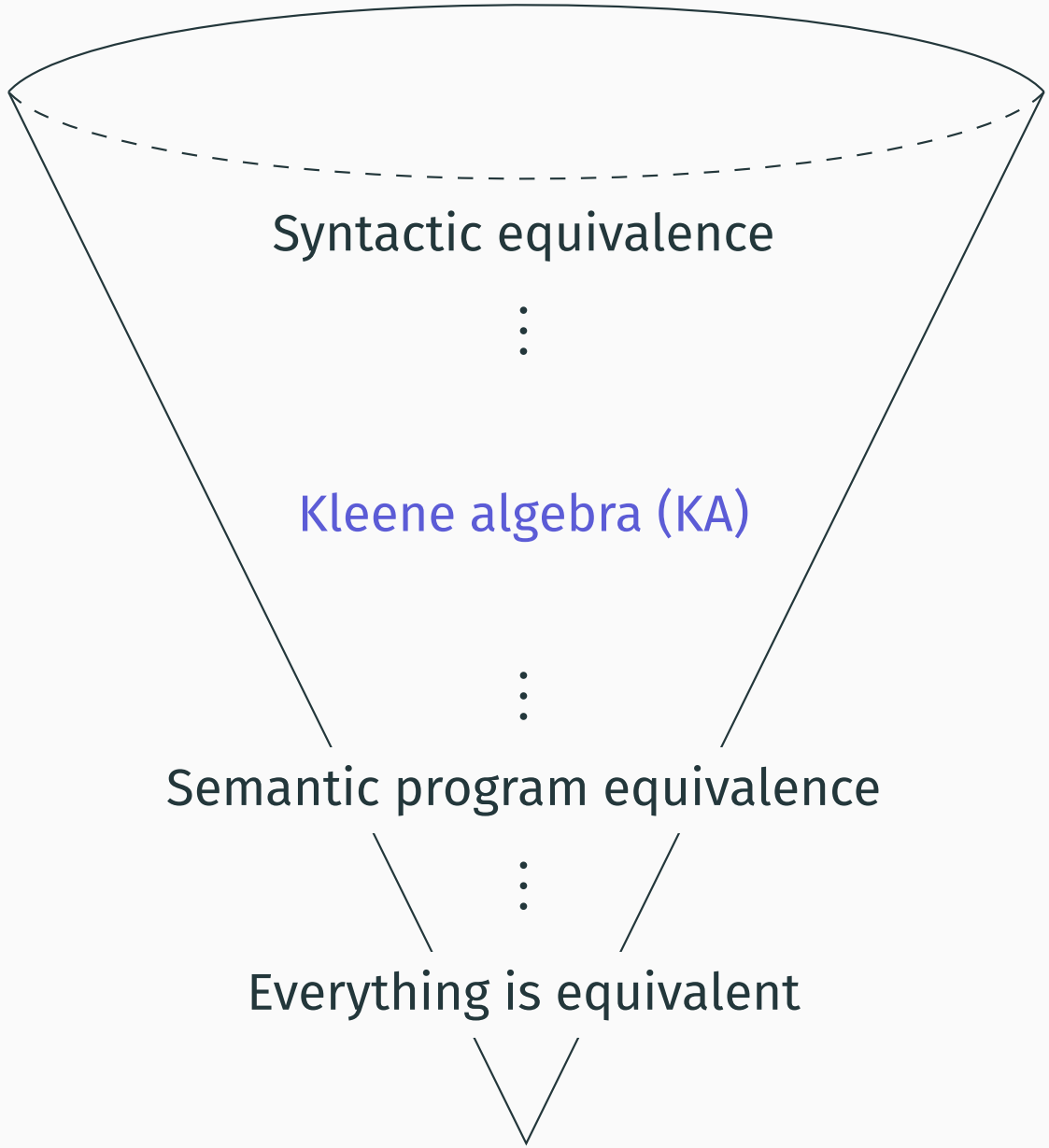

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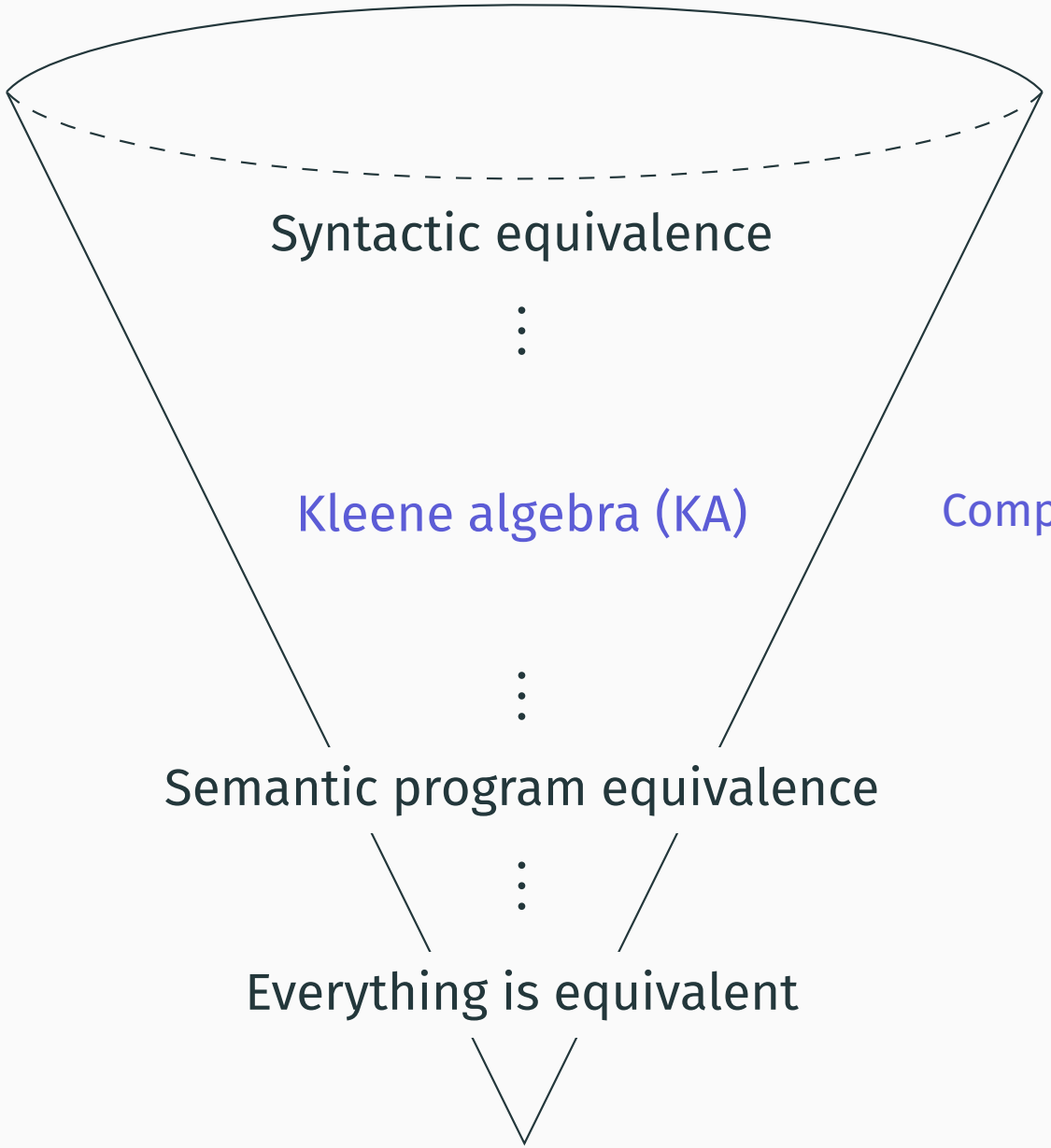

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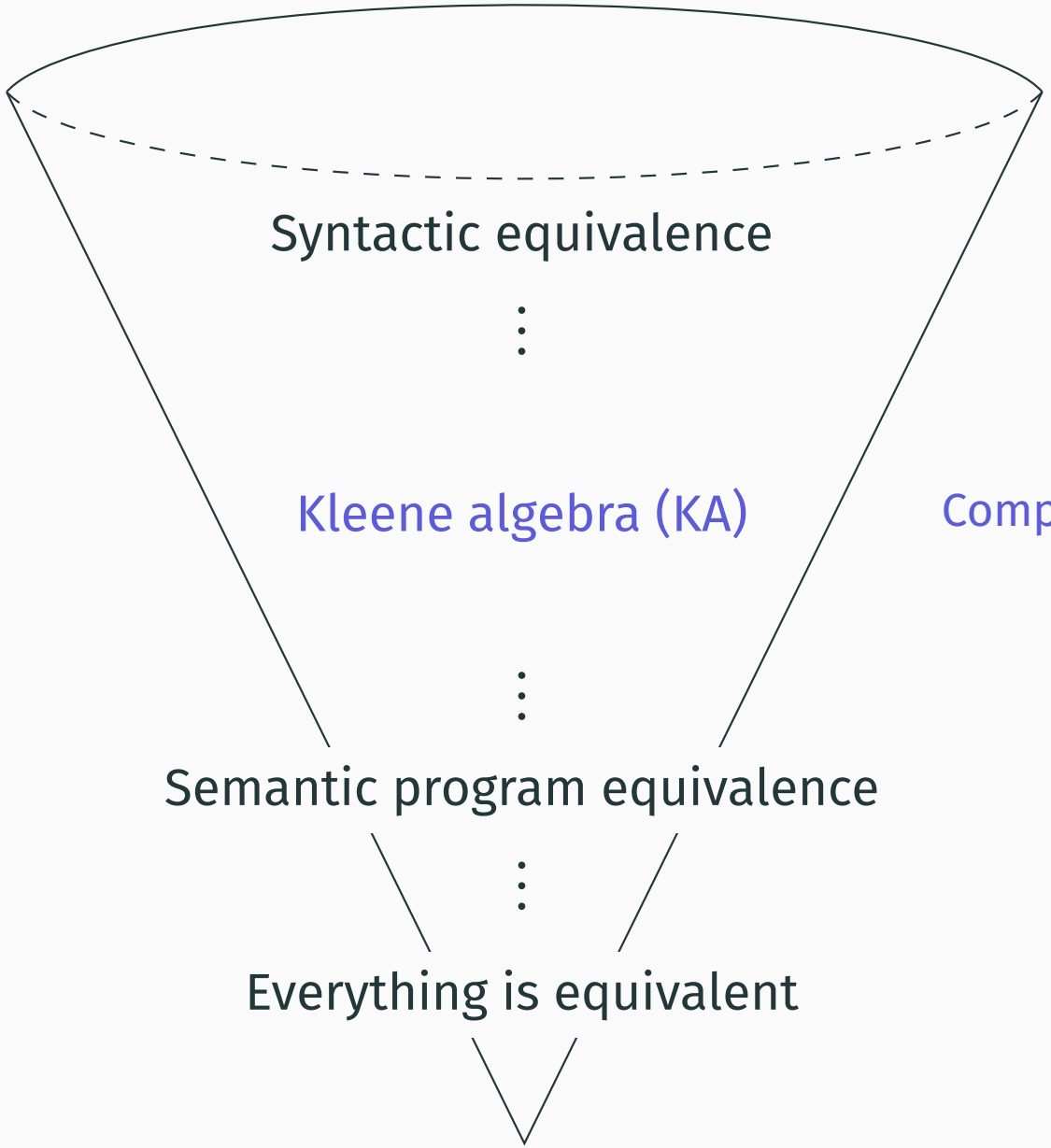

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Regex ignore if operations can be...

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Approach 1: add “features” to KA booleans, concurrency...

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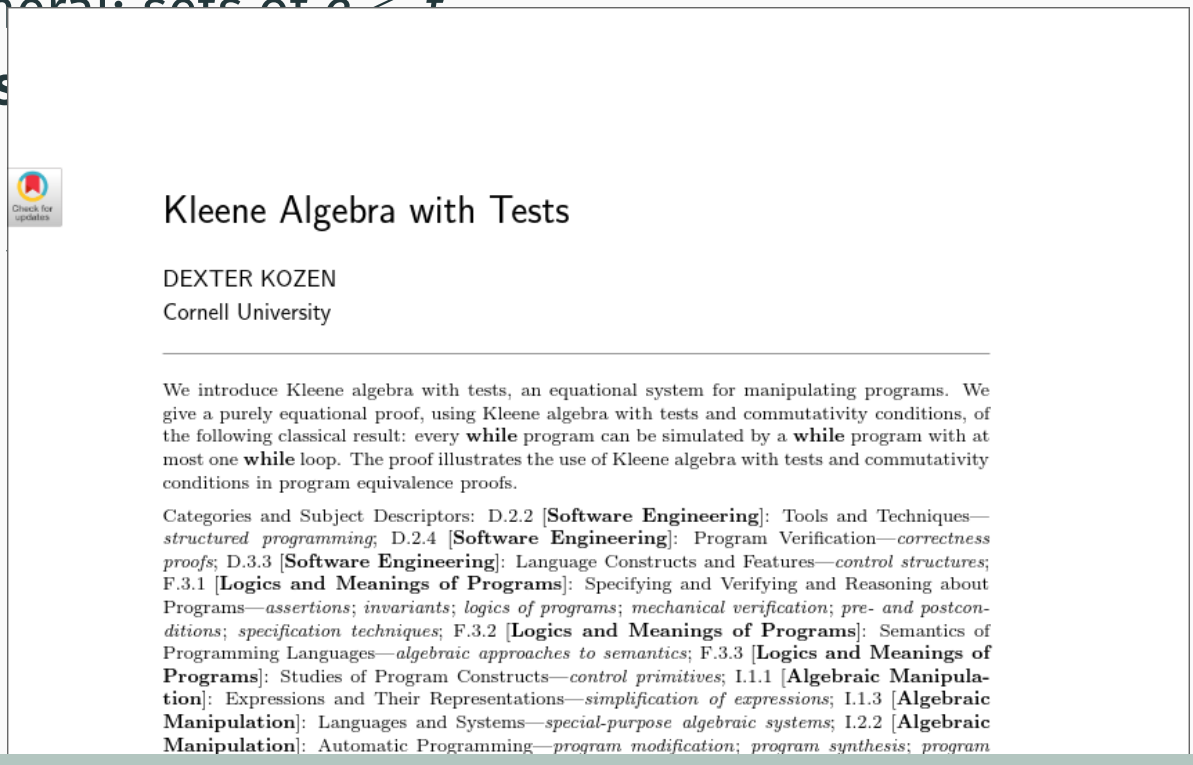
Approach



Kleene Algebra with Tests

DEXTER KOZEN
Cornell University

We introduce Kleene algebra with tests, an equational system for manipulating programs. We give a purely equational proof, using Kleene algebra with tests and commutativity conditions, of the following classical result: every **while** program can be simulated by a **while** program with at most one **while** loop. The proof illustrates the use of Kleene algebra with tests and commutativity conditions in program equivalence proofs.

Categories and Subject Descriptors: D.2.2 [**Software Engineering**]: Tools and Techniques—*structured programming*; D.2.4 [**Software Engineering**]: Program Verification—*correctness proofs*; D.3.3 [**Software Engineering**]: Language Constructs and Features—*control structures*; F.3.1 [**Logics and Meanings of Programs**]: Specifying and Verifying and Reasoning about Programs—*assertions*; *invariants*; *logics of programs*; *mechanical verification*; *pre- and postconditions*; *specification techniques*; F.3.2 [**Logics and Meanings of Programs**]: Semantics of Programming Languages—*algebraic approaches to semantics*; F.3.3 [**Logics and Meanings of Programs**]: Studies of Program Constructs—*control primitives*; I.1.1 [**Algebraic Manipulation**]: Expressions and Their Representations—*simplification of expressions*; I.1.3 [**Algebraic Manipulation**]: Languages and Systems—*special-purpose algebraic systems*; I.2.2 [**Algebraic Manipulation**]: Automatic Programming—*program modification*; *program synthesis*; *program*

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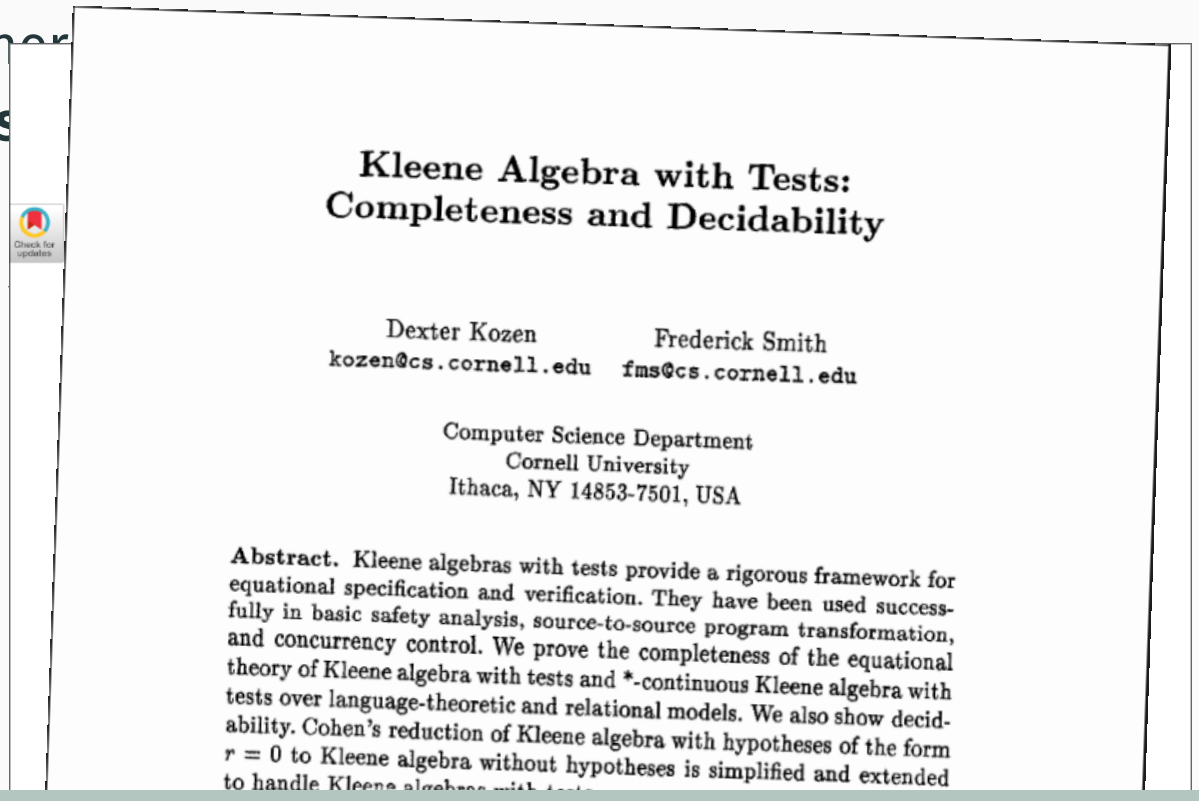

Kleene Algebra with Tests:
Completeness and Decidability

Dexter Kozen
kozen@cs.cornell.edu

Frederick Smith
fms@cs.cornell.edu

Computer Science Department
Cornell University
Ithaca, NY 14853-7501, USA

Abstract. Kleene algebras with tests provide a rigorous framework for equational specification and verification. They have been used successfully in basic safety analysis, source-to-source program transformation, and concurrency control. We prove the completeness of the equational theory of Kleene algebra with tests and *-continuous Kleene algebra with tests over language-theoretic and relational models. We also show decidability. Cohen's reduction of Kleene algebra with hypotheses of the form $r = 0$ to Kleene algebra without hypotheses is simplified and extended

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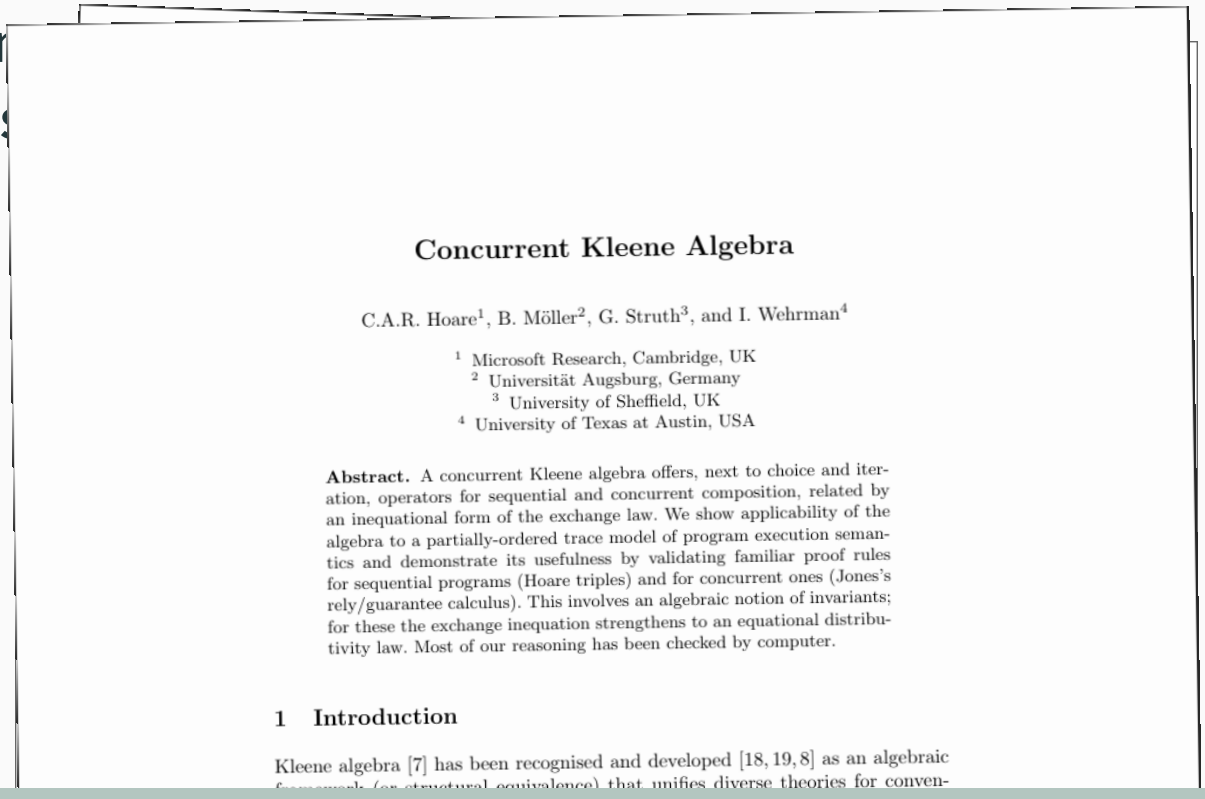

Concurrent Kleene Algebra

C.A.R. Hoare[1], B. Möller[2], G. Struth[3], and I. Wehrman[4]

[1] Microsoft Research, Cambridge, UK
[2] Universität Augsburg, Germany
[3] University of Sheffield, UK
[4] University of Texas at Austin, USA

Abstract. A concurrent Kleene algebra offers, next to choice and iteration, operators for sequential and concurrent composition, related by an inequational form of the exchange law. We show applicability of the algebra to a partially-ordered trace model of program execution semantics and demonstrate its usefulness by validating familiar proof rules for sequential programs (Hoare triples) and for concurrent ones (Jones's rely/guarantee calculus). This involves an algebraic notion of invariants; for these the exchange inequation strengthens to an equational distributivity law. Most of our reasoning has been checked by computer.

1 Introduction

Kleene algebra [7] has been recognised and developed [18, 19, 8] as an algebraic

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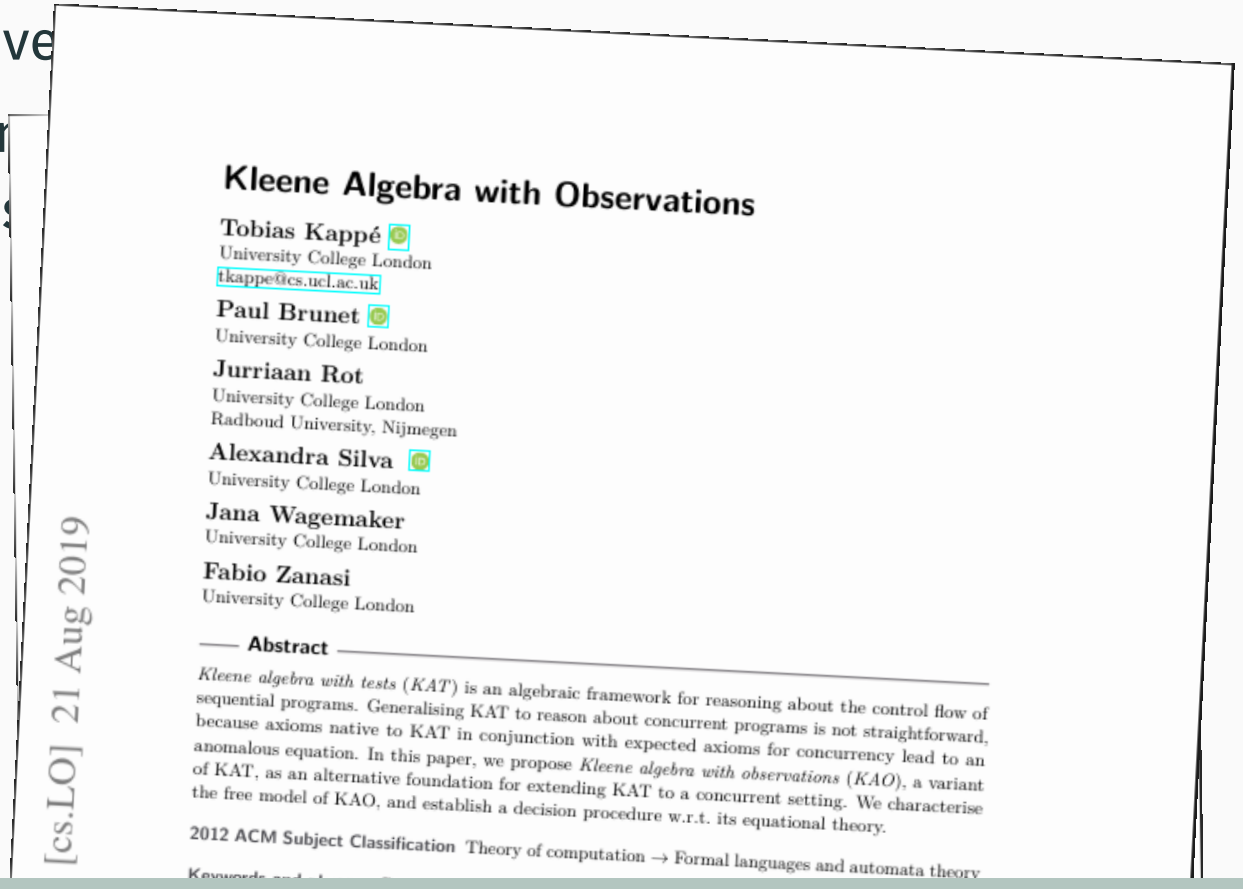

[cs.LO] 21 Aug 2019

Kleene Algebra with Observations

Tobias Kappé
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Paul Brunet
University College London

Jurriaan Rot
University College London
Radboud University, Nijmegen

Alexandra Silva
University College London

Jana Wagemaker
University College London

Fabio Zanasi
University College London

Abstract

Kleene algebra with tests (*KAT*) is an algebraic framework for reasoning about the control flow of sequential programs. Generalising KAT to reason about concurrent programs is not straightforward, because axioms native to KAT in conjunction with expected axioms for concurrency lead to an anomalous equation. In this paper, we propose *Kleene algebra with observations* (*KAO*), a variant of KAT, as an alternative foundation for extending KAT to a concurrent setting. We characterise the free model of KAO, and establish a decision procedure w.r.t. its equational theory.

2012 ACM Subject Classification Theory of computation → Formal languages and automata theory

Keywords and

“Propositional” Equivalence

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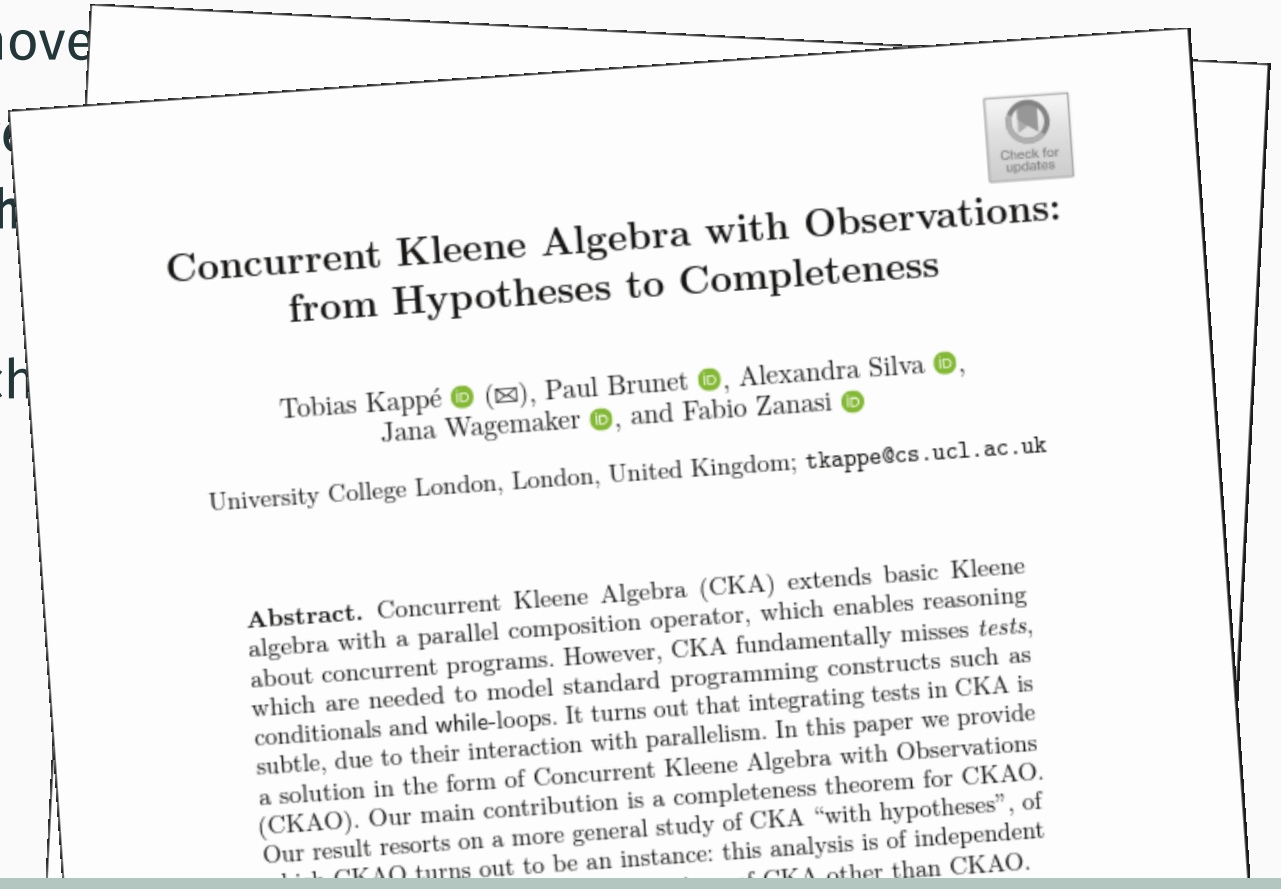

Concurrent Kleene Algebra with Observations: from Hypotheses to Completeness

Tobias Kappé (✉), Paul Brunet, Alexandra Silva, Jana Wagemaker, and Fabio Zanasi

University College London, London, United Kingdom; tkappe@cs.ucl.ac.uk

Abstract. Concurrent Kleene Algebra (CKA) extends basic Kleene algebra with a parallel composition operator, which enables reasoning about concurrent programs. However, CKA fundamentally misses *tests*, which are needed to model standard programming constructs such as conditionals and while-loops. It turns out that integrating tests in CKA is subtle, due to their interaction with parallelism. In this paper we provide a solution in the form of Concurrent Kleene Algebra with Observations (CKAO). Our main contribution is a completeness theorem for CKAO. Our result resorts on a more general study of CKA “with hypotheses”, of

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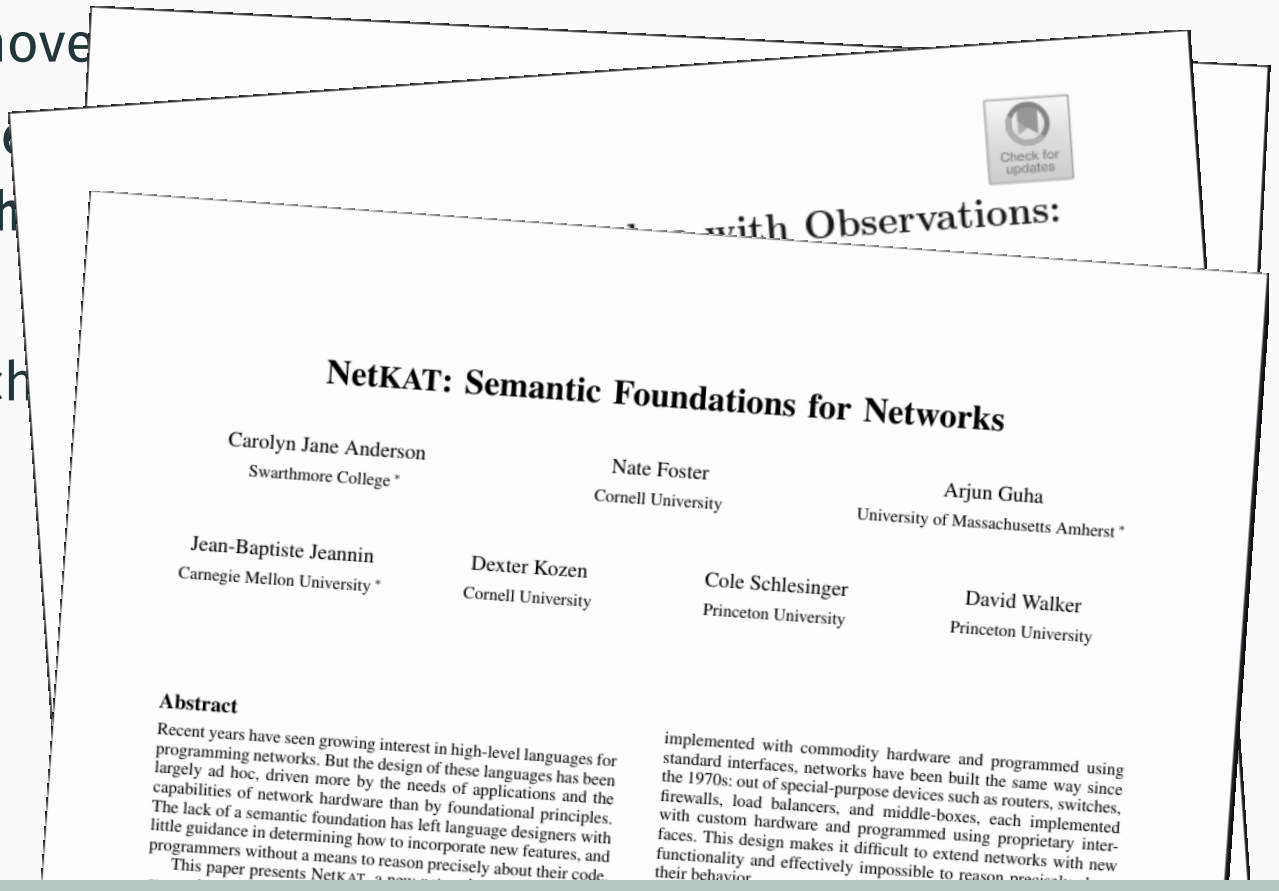

Check for updates

with Observations:

NetKAT: Semantic Foundations for Networks

Carolyn Jane Anderson
Swarthmore College *

Nate Foster
Cornell University

Arjun Guha
University of Massachusetts Amherst *

Jean-Baptiste Jeannin
Carnegie Mellon University *

Dexter Kozen
Cornell University

Cole Schlesinger
Princeton University

David Walker
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Abstract

Recent years have seen growing interest in high-level languages for programming networks. But the design of these languages has been largely ad hoc, driven more by the needs of applications and the capabilities of network hardware than by foundational principles. The lack of a semantic foundation has left language designers with little guidance in determining how to incorporate new features, and programmers without a means to reason precisely about their code. This paper presents NetKAT, a new

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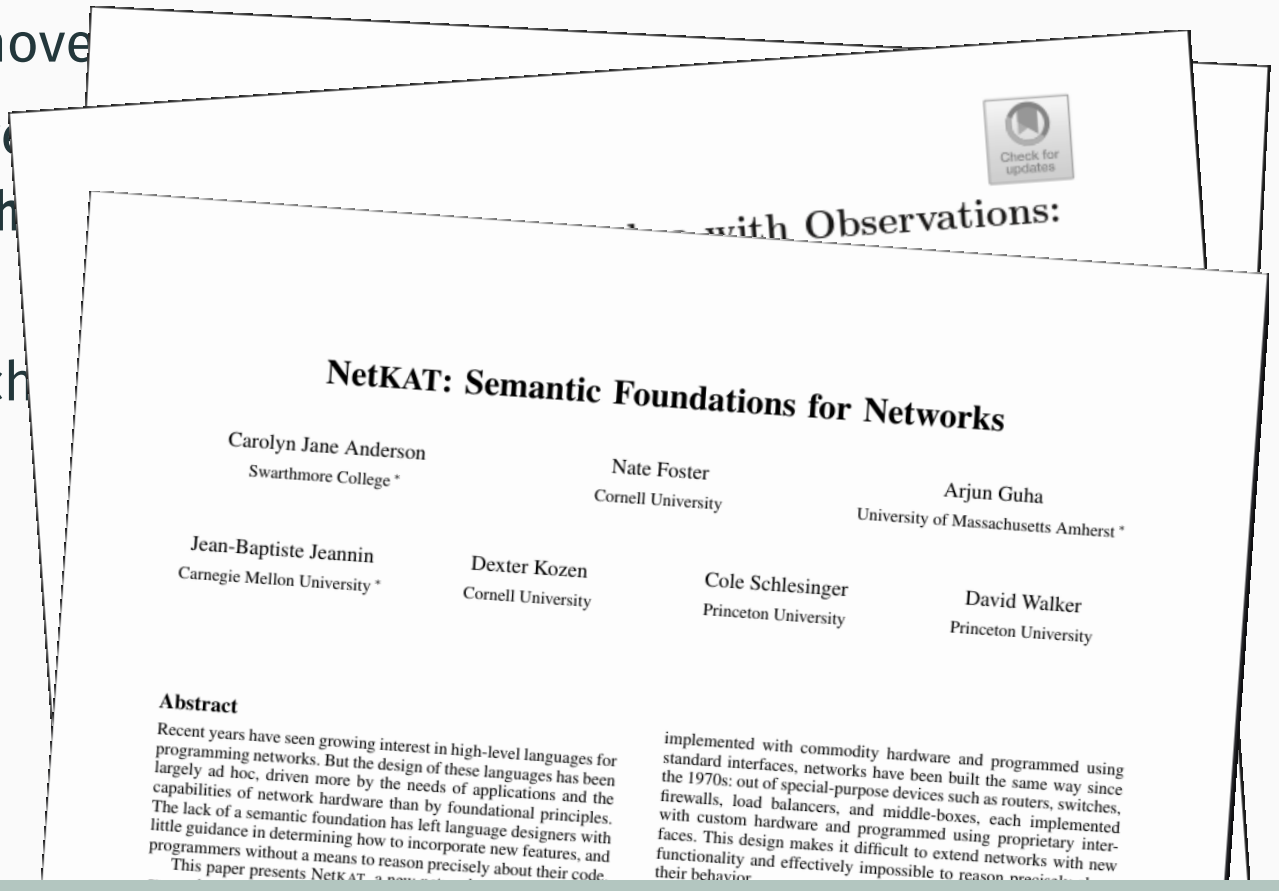

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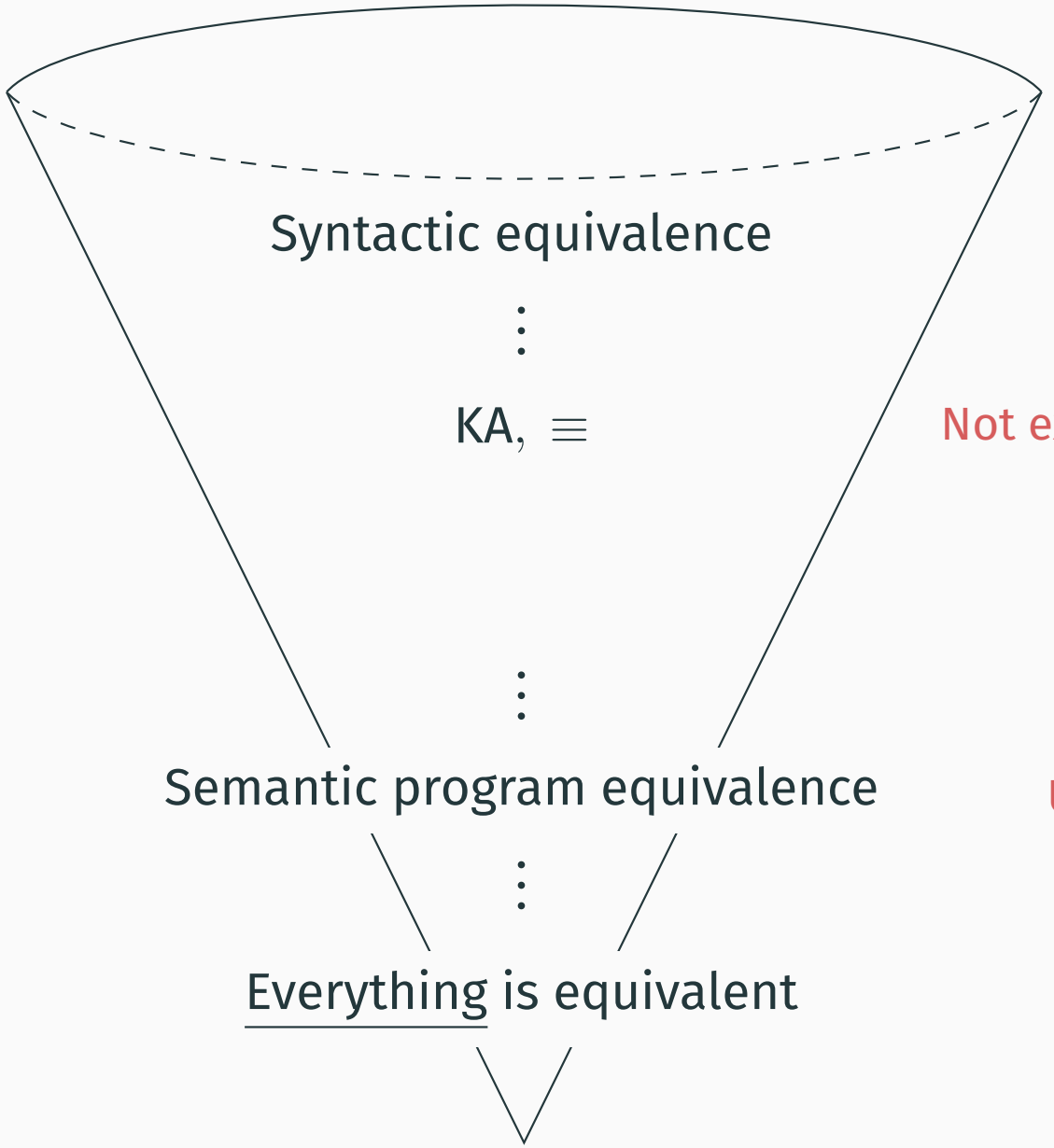
Approach 1: add “features” to KA booleans, concurrency...

Approach 2: add ~~axioms~~ “hypotheses” to KA directly $\rightarrow$ KA $+ H$

A Top-Down View on Program Equivalence Semantics



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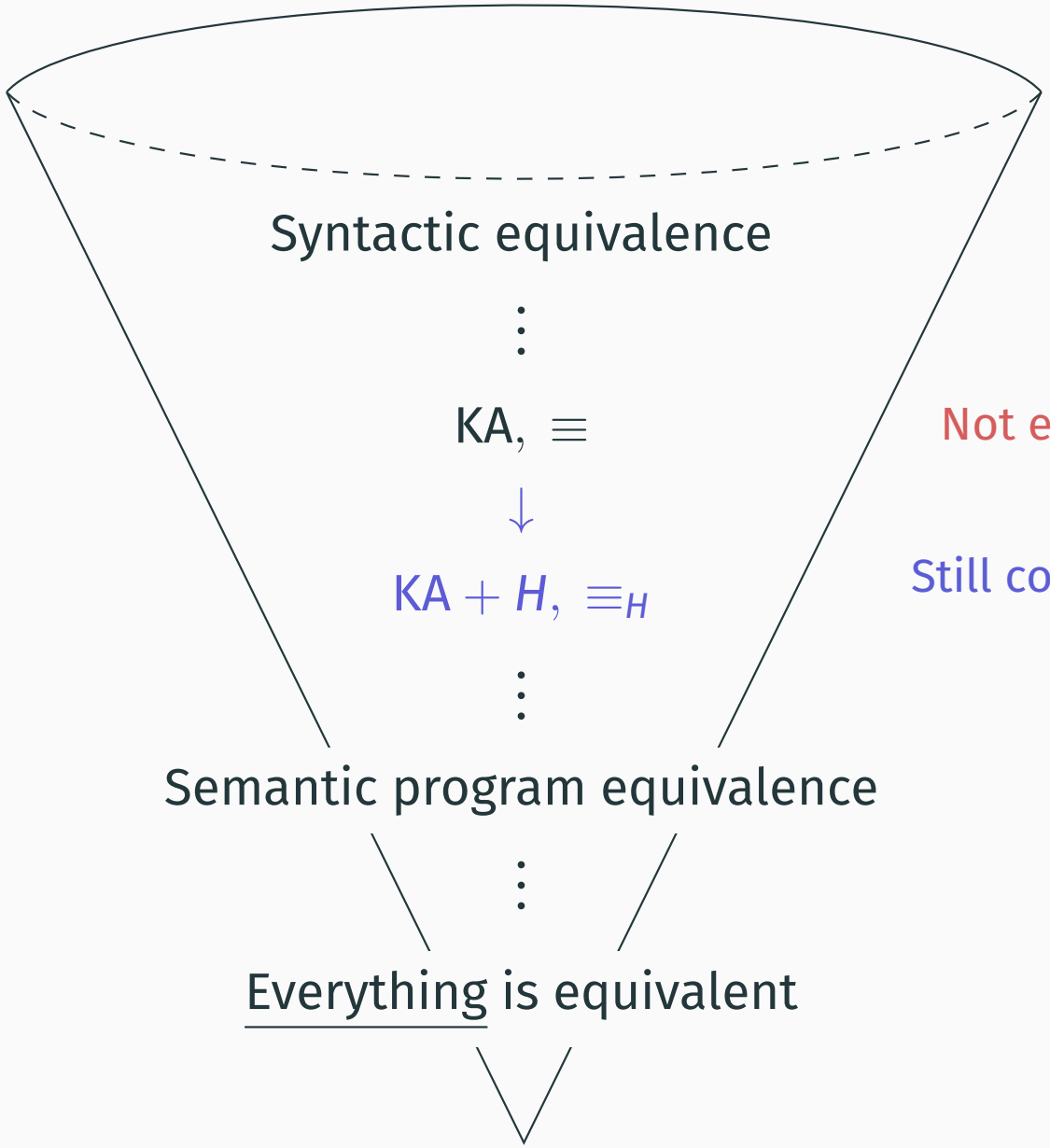

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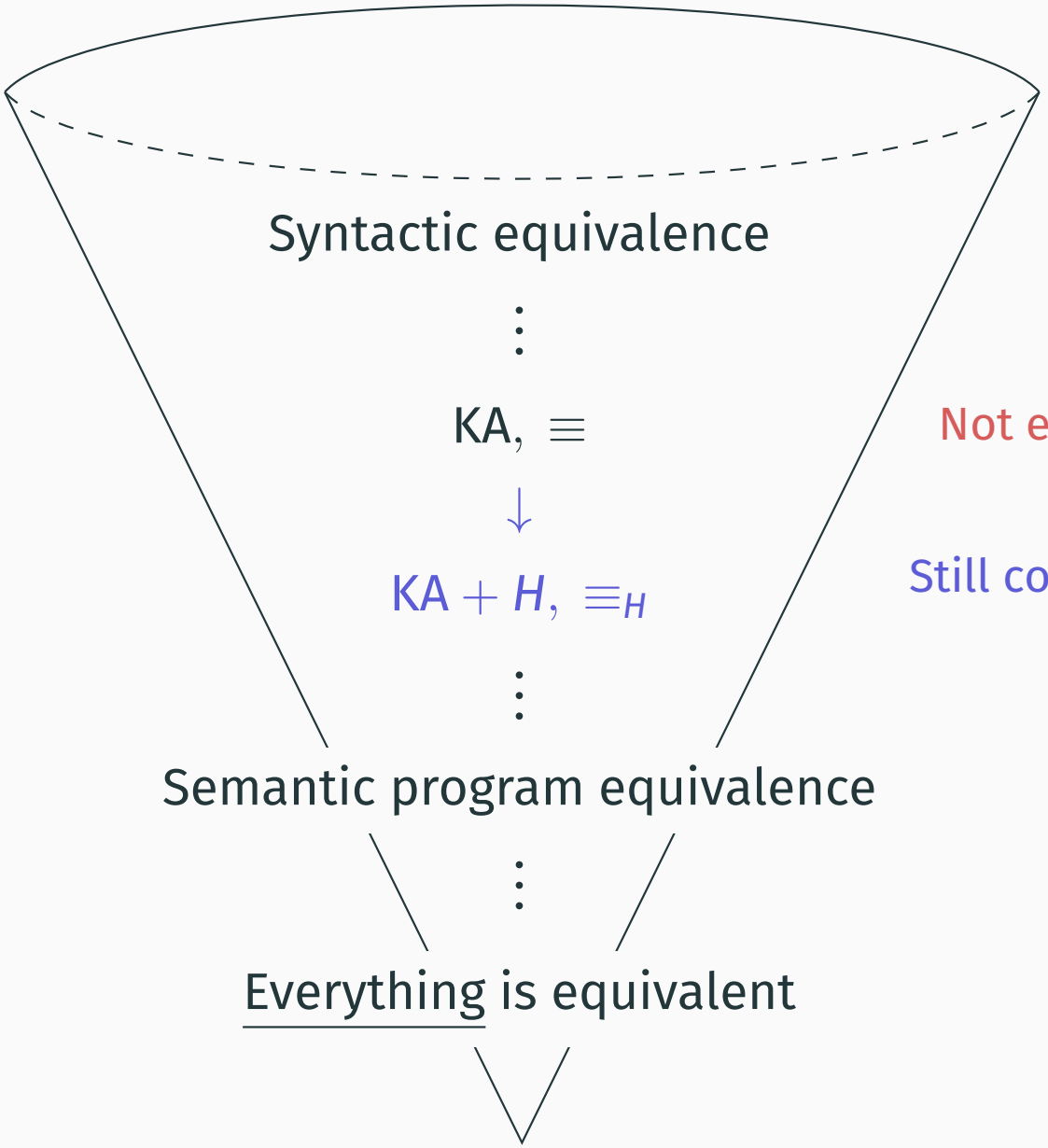

Hypothesis Closure

cf. Doumane et al. 2019

Definition

The *hypothesis closure* $H^*(L)$ for $e \leq w$ is L closed under:

$$L \mapsto L \cup \{ uw_e v : uwv \in L, w_e \in [\![e]\!] \}$$

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For example: $H = \{\mathtt{a}\mathtt{a} \leq \mathtt{a}\}$.

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$$[\![-]\!]_H := H^*([\![-]\!])$$

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Now $f \equiv_H g \;\Rightarrow\; [\![f]\!]_H = [\![g]\!]_H$. (soundness)

Game Plan

How to get completeness/decidability of KA $+$ H from KA?

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REDUCTION TODO LIST:

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Our contribution: a procedure for defining generic reductions, and a proof that when the procedure terminates (📌), it is correct.

Reduced Expressions via Automata

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$g \to A_g$ $r(g)$

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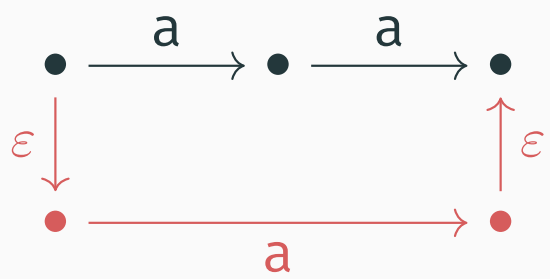

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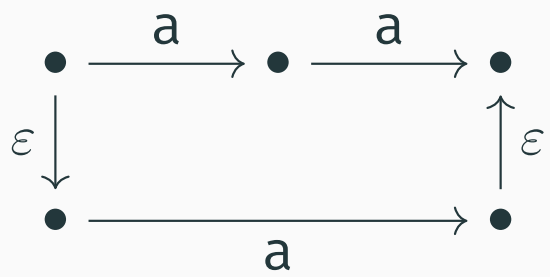

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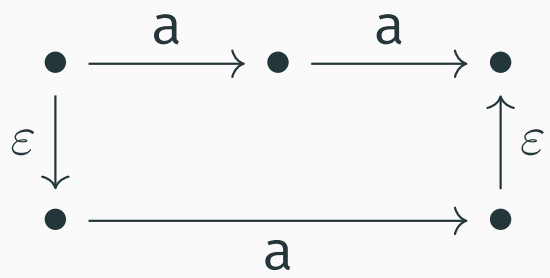

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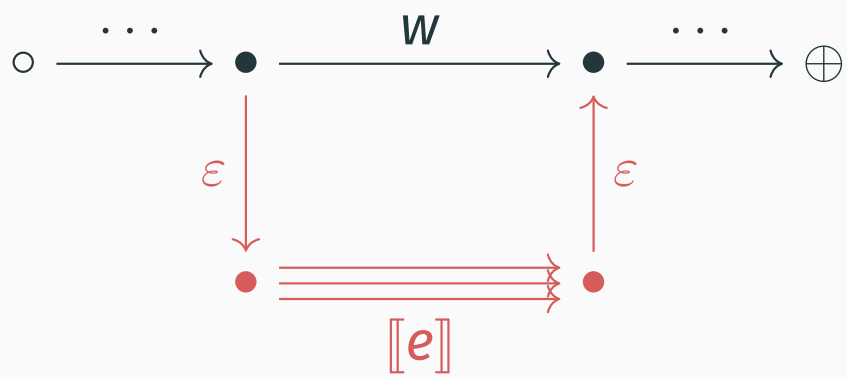

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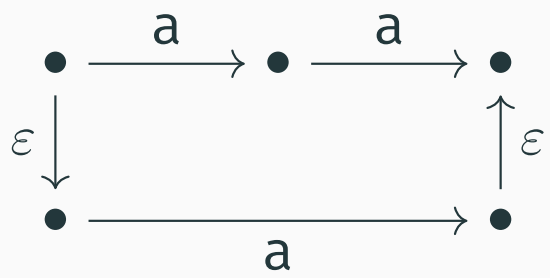

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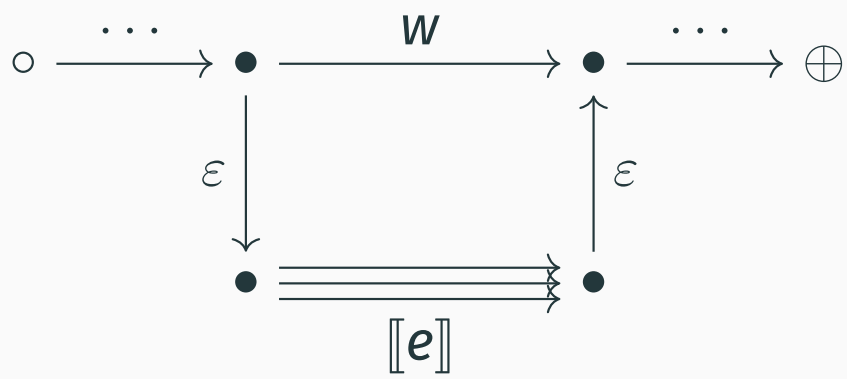

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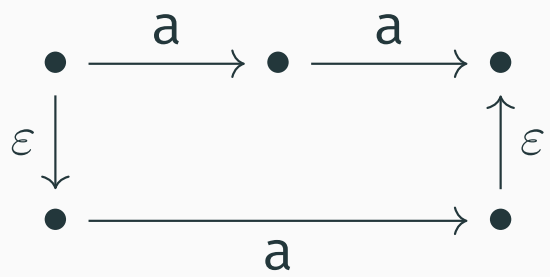

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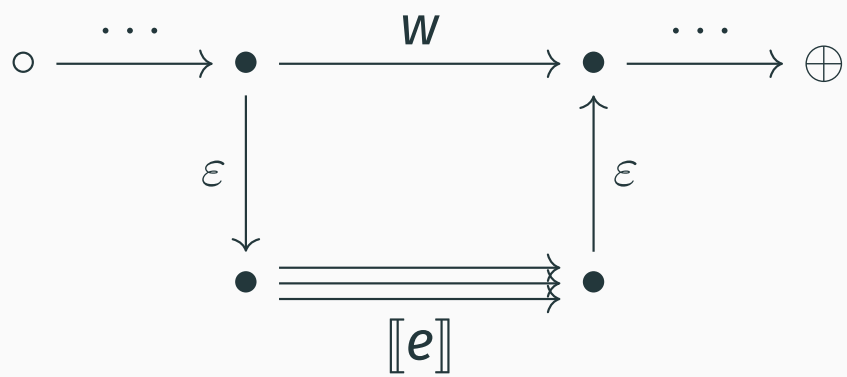

Various versions of this are well-known as strategies for “Approach 1”: see Book and Otto 1993; Kozen and Mamouras 2014; Pous, Rot, and Wagemaker 2024 for more.

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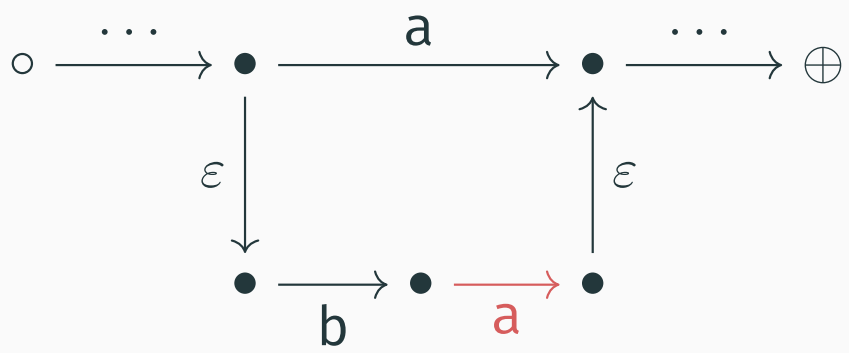

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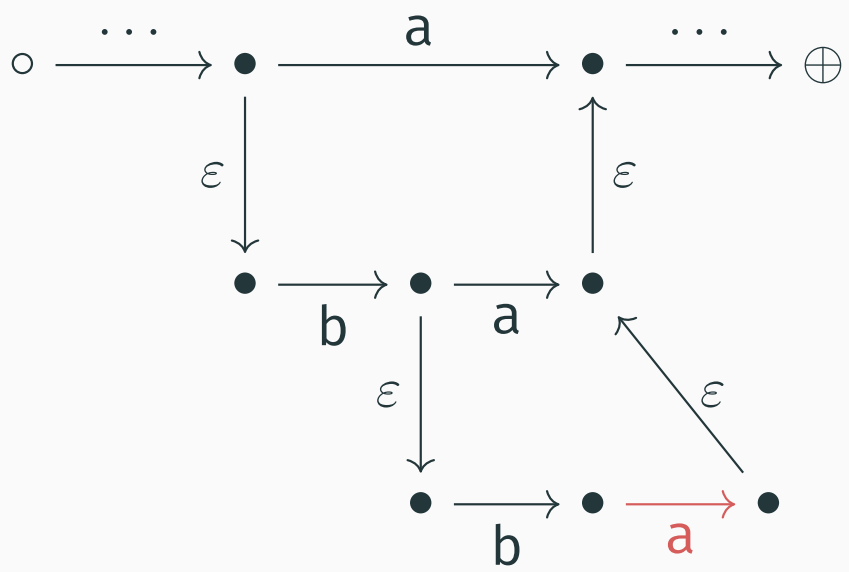

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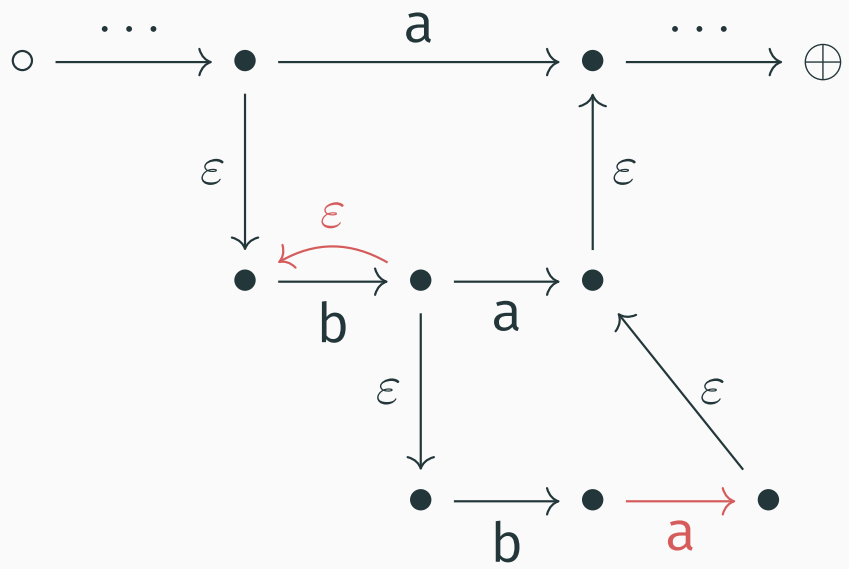

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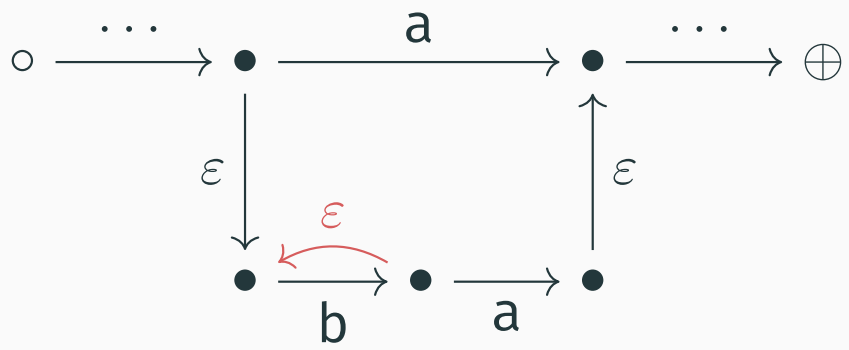

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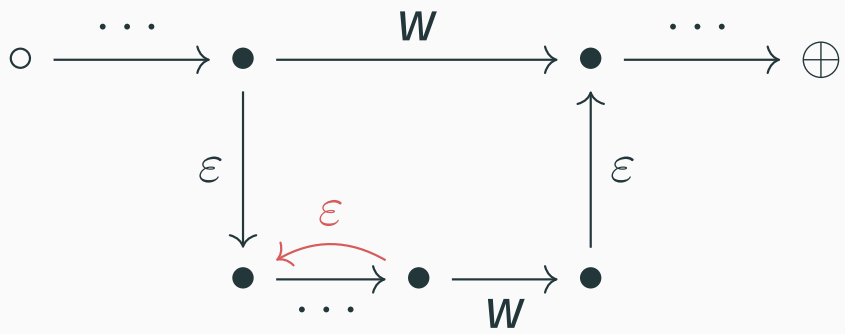

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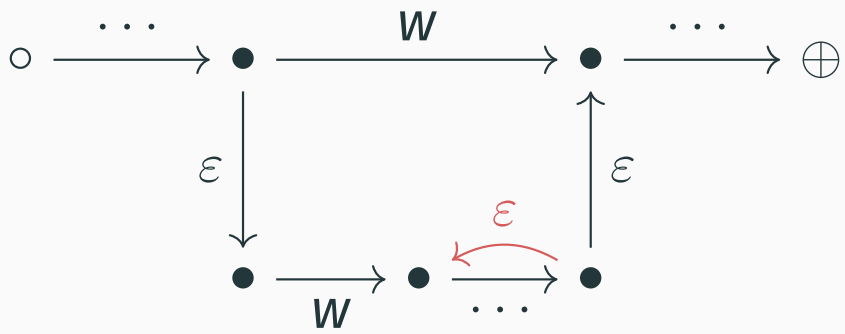

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Symmetric if w is a prefix as well.

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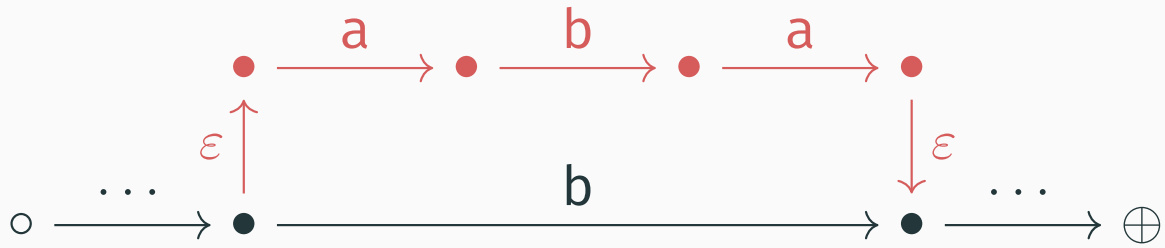

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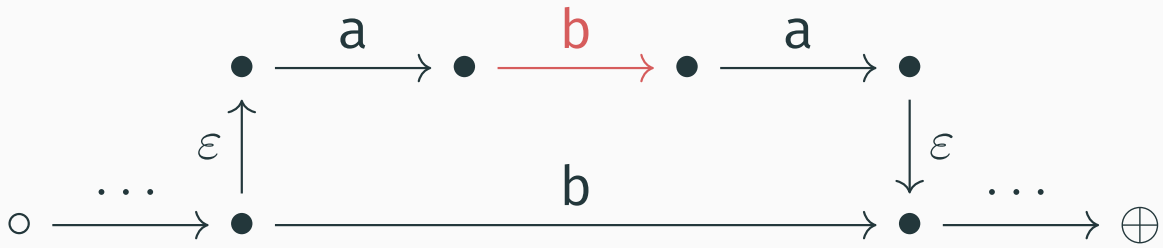

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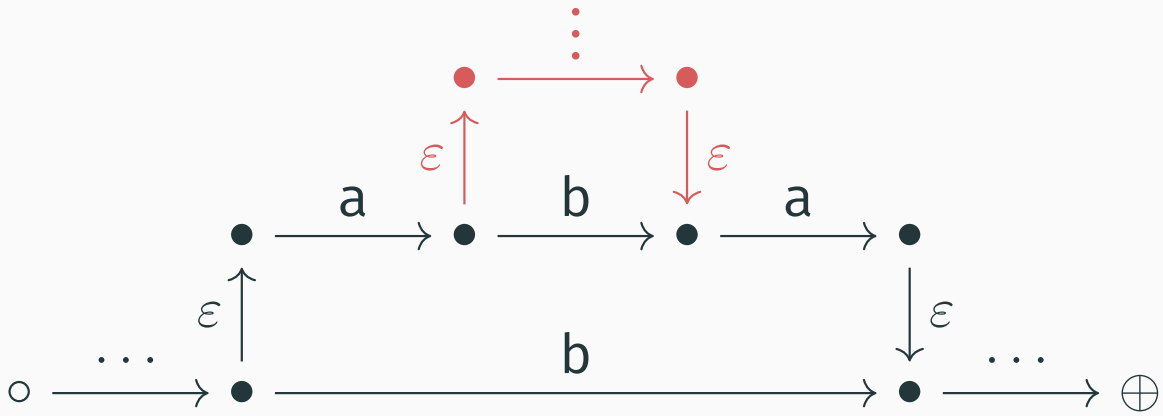

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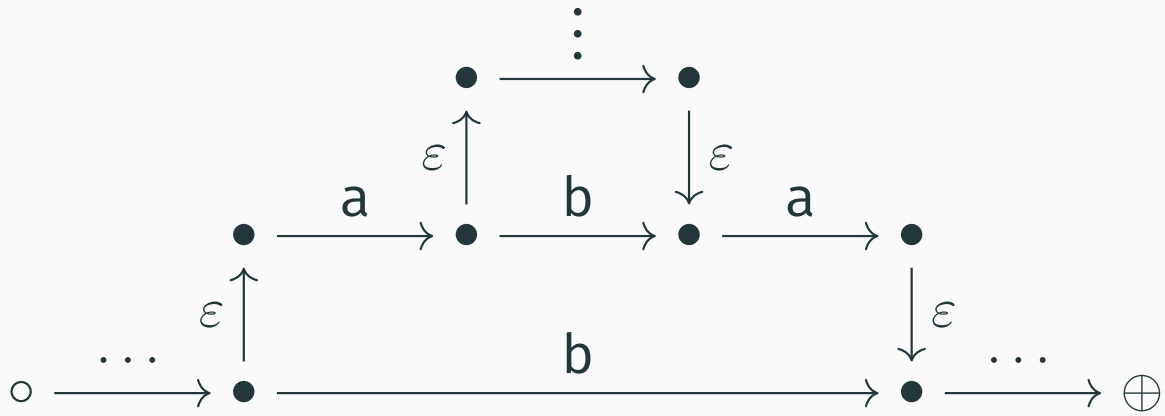

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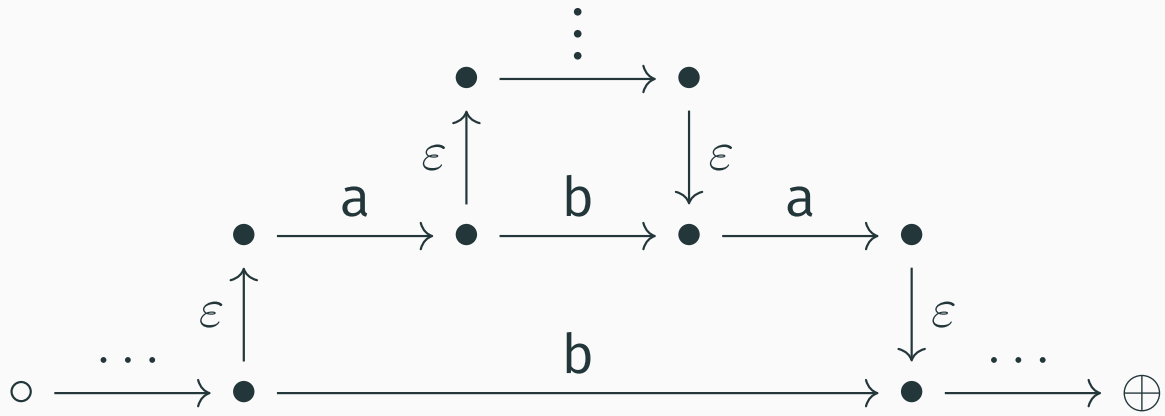

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Invariance (up to H) at each step of automaton “solutions”

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Next Steps

Possible applications:

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Other future work:

- Does T_H terminate when it can?
- General hypotheses $e \leq f$
- Sets of hypotheses

Thank you!

This paper on arxiv:

These slides on my website:

References

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